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Jet fields, connections and second-order differential equations

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Abstract. Jet fields are defined on fibred manifolds and represented as type (1, 1) tensor fields corresponding to Cartan-Ehresmann connections. When defined on the first jet bundle they may be used to generalise the second-order differential equation fields used in theoretical mechanics and to extend several existing results to field theories.

1. Introduction

The study of second-order differential equation fields is central to the theory of time-dependent Lagrangian particle dynamics (see, for example, Prince (1983), Prince and Eliezer (1980, 1981), Sarlet (1982) and Sarlet and Cantrijn (1981) to list a few recent works on the subject). In general, these fields may be defined as vector fields Λ on the manifold $TE \times \boldsymbol{R}$ (where E is the configuration manifold) which both annihilate all the contact forms and also satisfy the normalisation condition $\Lambda(t) = 1$ (Crampin 1977). In local coordinates $(t, q^\alpha, \dot{q}^\alpha)$ such a field would be written

$$\Lambda = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \Lambda^\alpha \frac{\partial}{\partial \dot{q}^\alpha} \tag{1}$$

where Λ^α are functions on $TE \times \boldsymbol{R}$.

The main argument of this paper is that it is profitable to consider, not the vector field Λ as such, but the associated type (1, 1) tensor field $\Lambda \otimes \mathrm{d}t$. The reason for introducing this apparent complexity is that there is a natural construction of the tensor field which generalises immediately to jet bundles, and may therefore be used in the study of Lagrangian field theories. The tensor field itself is essentially a Cartan-Ehresmann connection (Mangiarotti and Modugno 1983), but a definition in terms of 'jet fields' clarifies the analogy with mechanics and motivates this use of second-order jet fields. To demonstrate the use of the construction we prove three theorems which generalise results from the modern theory of tangent bundle geometry. Although we shall only consider first-order Lagrangian systems in the present work, the ideas may in principle be extended to higher-order systems.

The structure of this paper is as follows. In § 2 we summarise the notation used, and in § 3 we give a general definition of jet fields and develop some of their properties. Finally in § 4 we apply these ideas to second-order differential equation fields.

2. Notation

We adopt the conventions of Saunders (1987); in particular, we work with a locally trivial fibred manifold (E, π, M) and its first jet bundle $(J^1\pi, \pi_1, M)$ using local

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coordinates x^i, u^α and u_i^α ($1 \leq i \leq m$, $1 \leq \alpha \leq n$). M will be orientable with volume form $\Omega = dx^1 \wedge \ldots \wedge dx^m$, and $\mathcal{V}(\pi)$ will denote the space of vector fields on E vertical over M. We shall also require the theory of derivations established by Frölicher and Nijenhuis (1956) and we make use of the bracket operation on vector-valued forms defined in terms of the corresponding derivations of type d_*: if X, Y are vector-valued r- and s-forms respectively, then $[X, Y]$ is the unique vector-valued $(r+s)$-form satisfying

$$d_{[X,Y]} = d_X\, d_Y - (-1)^{rs}\, d_Y\, d_X. \tag{2}$$

This bracket operation reduces to the ordinary Lie bracket of vector fields when $r = s = 0$ (see Crampin and Ibort (1987) for a fuller discussion). d_h and d_v will denote the horizontal and vertical differentials mapping r-forms on $J^1\pi$ to $(r+1)$-forms on $J^2\pi$ (Tulczyjew 1980) (where, however, d_h and d_v are defined on $J^\infty\pi$). Finally, a contact m-form on $J^1\pi$ will be any m-form θ in $\Lambda_1^m(\pi_1)$ satisfying $(j^1\phi)^*\theta = 0$ for every local section ϕ of π; note that this differs from the definition found in, for example, Rogers and Shadwick (1982) in that not every contact m-form can be expressed as $\sigma^i \wedge (\partial/\partial x^i \lrcorner\, \Omega)$ where the σ^i are contact 1-forms.

3. Jet fields

We intend to define a jet field as a section of the bundle $(J^1\pi, \pi_{1,0}, E)$ in an obvious analogy to the definition of a vector field (see also Goldschmidt and Sternberg 1973). To make this analogy more substantial we shall define the action of a 1-jet (and therefore of a jet field) on a function, and construct the equivalent tensor to a jet field. We shall define integral sections, and give conditions for their existence. We shall also characterise symmetries and infinitesimal symmetries of jet fields.

Definition 3.1. Given a 1-jet $j_p^1\phi \in J^1\pi$, the action of the jet on functions is the map $C^\infty(E) \to T^*_{\phi(p)}E$ defined by

$$j_p^1\phi[f] = \pi^*(d(\phi^* f)_p). \tag{3}$$

This action is well defined for different representatives of $j_p^1\phi$ because it depends only on the first derivatives of ϕ. It is clearly a generalisation of the action of a tangent vector on a function; the main difference is that the resulting entity is a cotangent vector lifted from the base manifold rather than a number. In coordinates, one has

$$j_p^1\phi[f] = \left(\frac{\partial f}{\partial x^i}\bigg|_{\phi(p)} + u_i^\alpha(j_p^1\phi)\, \frac{\partial f}{\partial u^\alpha}\bigg|_{\phi(p)} \right) dx^i_{\phi(p)}. \tag{4}$$

Definition 3.2. A jet field $X : E \to J^1\pi$ is a section of the bundle $\pi_{1,0}$. The action of X on functions is the map $C^\infty(E) \to \Lambda_0^1(\pi)$ defined by

$$Xf_{\phi(p)} = X(\phi(p))[f]. \tag{5}$$

If the coordinate representation of X is given by $X_i^\alpha = u_i^\alpha \circ X$ then the action of X on functions may be written in coordinates as

$$Xf = \left(\frac{\partial f}{\partial x^i} + X_i^\alpha\, \frac{\partial f}{\partial u^\alpha} \right) dx^i. \tag{6}$$

This action, when extended to forms by the rule $X(\mathrm{d}\theta) = -\mathrm{d}(X\theta)$, is a derivation of type d_* and suggests the following result.

Proposition 3.3. There is a bijective correspondence between the jet fields $X : E \to J^1\pi$ and the type $(1, 1)$ tensor fields T on E satisfying the conditions

(1) T is a projection operator of constant rank m;

(2) $T(V) = 0$ for every vector field $V \in \mathcal{V}(\pi)$.

Proof. We give an explicit proof, rather than relying on the characterisation of X as a derivation. So suppose the jet field X is given. We fix $a \in E$ and let ϕ be a local section of π satisfying $j^1_{\pi(a)}\phi = X(a)$. Definition 3.1 suggests that we should define an endomorphism of cotangent vectors in T^*_aE by $\pi^*\phi^*$, or equivalently an endomorphism of tangent vectors in T_aE by $\phi_*\pi_*$. This endomorphism depends only on the first derivatives of ϕ, and is therefore independent of the choice of ϕ; in coordinates it is expressed as

$$\xi^i \frac{\partial}{\partial x^i}\bigg|_a + \eta^\alpha \frac{\partial}{\partial u^\alpha}\bigg|_a \mapsto \xi^i \left(\frac{\partial}{\partial x^i}\bigg|_a + u^\alpha_i(X(a)) \frac{\partial}{\partial u^\alpha}\bigg|_a \right). \tag{7}$$

Taking this endomorphism at each $a \in E$ yields a type $(1, 1)$ tensor field $\tilde{X}$ which is seen to be smooth and to satisfy the conditions of the proposition from its coordinate representation

$$\tilde{X} = \left(\frac{\partial}{\partial x^i} + X^\alpha_i \frac{\partial}{\partial u^\alpha} \right) \otimes \mathrm{d}x^i. \tag{8}$$

Two distinct jet fields will have different coordinate representations at some point in E so that the corresponding tensor fields will differ; the correspondence is therefore injective. Furthermore, any tensor field T satisfying the conditions of the proposition must have a coordinate representation of the form

$$T = \left(\frac{\partial}{\partial x^i} + T^\alpha_i \frac{\partial}{\partial u^\alpha} \right) \otimes \mathrm{d}x^i \tag{9}$$

so locally there is a jet field with coordinates T^α_i which gives rise to T: on overlapping coordinate patches these local jet fields must agree since the correspondence is injective, so we can define a global jet field and hence the correspondence is also surjective.

It is clear from this result that, for a function f, $Xf = \mathrm{d}_{\tilde{X}}f$. A similar result applies to jet fields and tensor fields defined only on a given open subset of E; in fact, one should note that the above proposition actually makes no assertion about the global existence of either type of object. In general, $(J^1\pi, \pi_{1,0}, E)$ has the structure of an affine bundle which only takes the additional structure of a vector bundle in favourable circumstances. As a consequence, the addition of jet fields is normally undefined.

We now move on to consider integral sections of a jet field.

Definition 3.4. An integral section of the jet field X is a local section ϕ of π satisfying $j^1\phi = X \circ \phi$.

This definition clearly mimics the corresponding definition for an integral curve of a vector field. There is, however, an important difference: there is no guarantee that integral sections of a given jet field will exist, even locally. To see this, note that the definition of an integral section may be rephrased in coordinates as

$$\frac{\partial \phi^\alpha}{\partial x^i} = X_i^\alpha \circ \phi \tag{10}$$

and that this set of partial differential equations must satisfy an integrability condition before solutions ϕ^α can exist. In fact, the following result is essentially a translation of Frobenius' theorem into the language of jet fields.

Proposition 3.5. The jet field X has integral sections if, and only if, the Nijenhuis tensor of $\tilde{X}$ vanishes; such a jet field is termed integrable.

Proof. Recall that the Nijenhuis tensor $N_{\tilde{X}}$ of the type $(1, 1)$ tensor field $\tilde{X}$ is the type $(2, 1)$ tensor field defined by its action on pairs of vector fields:

$$N_{\tilde{X}}(U, V) = \tilde{X} \circ \tilde{X}[U, V] + [\tilde{X}(U), \tilde{X}(V)] - \tilde{X}[\tilde{X}(U), V] - \tilde{X}[U, \tilde{X}(V)]. \tag{11}$$

Consider this expression locally and let U, V be coordinate vector fields. Then $N_{\tilde{X}}(\partial/\partial u^\alpha, \partial/\partial u^\beta)$ and $N_{\tilde{X}}(\partial/\partial x^i, \partial/\partial u^\beta)$ vanish identically; the only non-trivial expression comes from

$$N_{\tilde{X}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left[\left(\frac{\partial X_j^\alpha}{\partial x^i} + X_i^\beta \frac{\partial X_j^\alpha}{\partial u^\beta}\right) - \left(\frac{\partial X_i^\alpha}{\partial x^j} + X_j^\beta \frac{\partial X_i^\alpha}{\partial u^\beta}\right)\right] \frac{\partial}{\partial u^\alpha}. \tag{12}$$

However, the vanishing of this expression is precisely the condition for equations (10) to be integrable in the sense of Frobenius.

We can also describe this condition in more geometric terms. When a diagonalisable type $(1, 1)$ tensor field is considered as an endomorphism of the tangent bundle, the vanishing of its Nijenhuis tensor implies that the distributions generated by the eigenspaces of the endomorphism are involutive. Now $\tilde{X}$ is a projection operator, and so has eigenvalues zero and one. The distribution corresponding to the eigenvalue zero just contains the vertical vectors, and is always involutive; its integral manifolds are the fibres of π. The distribution corresponding to the eigenvalue one may or may not be involutive; if it is, then the image sets of the integral sections will be its integral manifolds. In fact, this latter distribution may be considered as defining a bundle of 'horizontal' vectors on E, so (apart from the question of horizontal completeness) the jet field determines a Cartan–Ehresmann connection on π; the connection is flat exactly when the jet field is integrable. If π happens to be a principal fibre bundle and the horizontal subspaces are equivariant with respect to the group action then the jet field may be studied using the well established theory of gauge fields.

We next consider symmetries of jet fields. In the case of a vector field a symmetry may be regarded as a diffeomorphism of the manifold which permutes the integral sections without changing their parametrization. It therefore seems natural to consider diffeomorphisms of E which project onto the identity of M and which permute the integral sections of X. So suppose ψ is such a diffeomorphism. We wish to assert that ψ is a symmetry of X if, whenever ϕ is an integral section, so is $\psi \circ \phi$. Using the characterisation of $\tilde{X}$ as an endomorphism of $T_{\phi(p)}E$ for each p in the domain of ϕ, $\tilde{X}_{\phi(p)} = \phi_* \pi_*$, we find

$$\tilde{X}_{\psi\phi(p)} = (\psi_* \phi_*)(\pi_* \psi_*^{-1}) = \psi_* \tilde{X}_{\phi(p)} \psi_*^{-1}$$

or, more generally,

$$\tilde{X}_{\psi(a)} = \psi_* \tilde{X}_a \psi_*^{-1} \tag{13}$$

for $a \in E$. We are therefore led to the following definition, which makes sense whether or not X is integrable.

Definition 3.6. A symmetry of the jet field X is a diffeomorphism ψ of E which projects onto the identity transformation of M and which satisfies $\psi_* \tilde{X} = \tilde{X} \psi_*$ where $\tilde{X}$ is regarded as acting on tangent vectors.

Proposition 3.7. If X is integrable then ψ is a symmetry of X if, and only if, ψ permutes the integral sections of X.

In the infinitesimal version of this construction we consider a vertical vector field Γ on E, with flow ψ_t.

Definition 3.8. An infinitesimal symmetry of the jet field X is a vertical vector field Γ satisfying $\mathscr{L}_\Gamma \tilde{X} = 0$.

Proposition 3.9. If X is integrable then Γ is an infinitesimal symmetry of X if, and only if, for each t the diffeomorphism ψ_t permutes the integral sections of X.

Proof. For simplicity in the second part of the proof we suppose Γ to be complete, although this assumption is not necessary for the result to be valid.

First, suppose each ψ_t permutes the integral sections of X. Then by (3.6) and (3.7), $\psi_{t*} \tilde{X} = \tilde{X} \psi_{t*}$. Now for every vector field Y on E, and every $a \in E$,

$$\begin{aligned}(\mathscr{L}_\Gamma \tilde{X})(Y)_a &= \mathscr{L}_\Gamma(\tilde{X}(Y))_a - \tilde{X}_a(\mathscr{L}_\Gamma Y)_a \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{t*} \tilde{X}(Y)_{\psi_{-t}(a)} - \tilde{X}_a \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{t*} Y_{\psi_{-t}(a)} \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\psi_{t*} \tilde{X}_{\psi_{-t}(a)} Y_{\psi_{-t}(a)} - \tilde{X}_a \psi_{t*} Y_{\psi_{-t}(a)})\end{aligned} \tag{14}$$

using continuity of the endomorphism $\tilde{X}_a$ of T_aE. Consequently the right-hand side of this expression vanishes, and so $\mathscr{L}_\Gamma \tilde{X} = 0$.

The converse involves a careful proof which effectively integrates along the flow ψ_t. So suppose that $\mathscr{L}_\Gamma \tilde{X} = 0$. Then for every vector field Y on E, and every $a \in E$,

$$\mathscr{L}_\Gamma(\tilde{X}(Y))_a = \tilde{X}_a(\mathscr{L}_\Gamma Y)_a \tag{15}$$

which we may write as

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{t*} \tilde{X}_{\psi_{-t}(a)}(Y_{\psi_{-t}(a)}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \tilde{X}_a(\psi_{t*} Y_{\psi_{-t}(a)}). \tag{16}$$

Fix a and choose an arbitrary real number h, writing $a_{-h} = \psi_{-h}(a)$; then equation (16) is still true with a replaced by a_{-h}. For each tangent vector $\eta \in T_{a_{-h}}E$ there is certainly a smooth vector field Y satisfying, for sufficiently small t, $Y_{\psi_{-t}(a_{-h})} = \psi_{-t*}\eta$; with this choice of Y we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{t*}\tilde{X}_{\psi_{-t}(a_{-h})}\psi_{-t*}\eta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{X}_{a_{-h}}(\eta) = 0 \tag{17}$$

since $\tilde{X}_{a_{-h}}(\eta)$ is independent of t. Also, $\eta \in T_{a_{-h}}E$ is arbitrary, so we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{t*}\tilde{X}_{\psi_{-t}(a_{-h})}\psi_{-t*} = 0 \tag{18}$$

as an endomorphism of $T_{a_{-h}}E$.

Now premultiply this equation by ψ_{h*} and postmultiply by ψ_{-h*}. The result is an equation relating endomorphisms of T_aE, and if we write τ for $t+h$ we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=h} \psi_{\tau*}\tilde{X}_{\psi_{-\tau}(a)}\psi_{-\tau*} = 0. \tag{19}$$

However h, too, is arbitrary and so we find that $\psi_{\tau*}\tilde{X}_{\psi_{-\tau}(a)}\psi_{-\tau*}$ is independent of τ and therefore equals its value when τ is zero:

$$\psi_{\tau*}\tilde{X}_{\psi_{-\tau}(a)}\psi_{-\tau*} = \tilde{X}_a. \tag{20}$$

The result follows.

Note that if the coordinate representation of the vertical vector field Γ is $\Gamma^\alpha\,\partial/\partial u^\alpha$, then the condition for Γ to be an infinitesimal symmetry of X takes the 'Lax form'

$$\frac{\partial\Gamma^\beta}{\partial x^i} = \Gamma^\alpha \frac{\partial X_i^\beta}{\partial u^\alpha} - X_i^\alpha \frac{\partial \Gamma^\beta}{\partial u^\alpha}. \tag{21}$$

As a final remark in this section, we observe that when the base manifold M is one-dimensional with volume form $\mathrm{d}t$, we may consider the vector fields Y on E which are π-related to the vector field $\partial/\partial t$ on M. Any two such vector fields Y differ by a vertical vector field, and so the contraction $\tilde{X}(Y)$ does not depend on the particular choice of Y. We may therefore write it as $\tilde{X}(\partial/\partial t)$, and in this way recover the representation of X as a vector field rather than a tensor field. When $\dim M > 1$ this representation is not available.

4. Second-order jet fields

The second jet manifold $J^2\pi$ is canonically embedded in $J^1\pi_1$ by a map $\iota_{1,1}$; a second-order jet field is a jet field on the manifold $J^1\pi$ which satisfies additional conditions to ensure that its image is contained within $\iota_{1,1}(J^2\pi)$.

Definition 4.1. A section $X: J^1\pi \to J^1\pi_1$ of $(\pi_1)_{1,0}$ is called a second-order jet field if, for every contact m-form θ on $J^1\pi$, $i_{\tilde{X}}\theta = (m-1)\theta$.

In general, the coordinate representation of the tensor field $\tilde{X}$ on $J^1\pi$ will be

$$\tilde{X}=\left(\frac{\partial}{\partial x^i}+X_i^\alpha\frac{\partial}{\partial u^\alpha}+X_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}\right)\otimes \mathrm{d}x^i \tag{22}$$

but by considering the contact m-forms $(\mathrm{d}u^\beta-u_k^\beta\,\mathrm{d}x^k)\wedge(\partial/\partial x^l\lrcorner\Omega)$ and $\mathrm{d}u_k^\beta\wedge(\partial/dx^i\lrcorner\Omega)-\mathrm{d}u_i^\beta\wedge(\partial/\partial x^k\lrcorner\Omega)$ and applying definition 4.1 we find that $X_i^\alpha=u_i^\alpha$ and $X_{ij}^\alpha=X_{ji}^\alpha$. Consequently $u_{i;}^\alpha\circ X=u_{;i}^\alpha\circ X$ and $u_{i;j}^\alpha\circ X=u_{j;i}^\alpha\circ X$ where the functions with semicolon subscripts are coordinates on $J^1\pi_1$. These are just the conditions for $\mathrm{Im}(X)$ to be contained in $\iota_{1,1}(J^2\pi)$, so we may regard the second-order jet field X as a section of $\pi_{2,1}$; furthermore, every section of $\pi_{2,1}$ arises in this way.

According to definition 3.4, integral sections of X are sections ψ of the bundle $(J^1\pi,\pi_1,M)$ satisfying $j^1\psi=X\circ\psi$; the following lemma gives a more useful characterisation.

Lemma 4.2. If X is a second-order jet field then any integral section ψ of X is the 1-jet extension of a section of π.

Proof. If σ is any contact 1-form on $J^1\pi$ then $\tilde{X}(\sigma)=0$; this may readily be seen using the coordinate representation of $\tilde{X}$. Therefore $\pi_1^*\psi^*\sigma=0$, and since π_1^* is injective we have $\psi^*\sigma=0$. Since σ is arbitrary ψ must equal $j^1\phi$, where $\phi=\pi_{1,0}\circ\psi$ is a section of π.

As a result of this lemma we may regard ϕ as the integral section rather than $j^1\phi$. Consequently integral sections of X satisfy the equations

$$\frac{\partial^2\phi^\alpha}{\partial x^i\,\partial x^j}=X_{ij}^\alpha\circ j^1\phi \tag{23}$$

or, in a more familiar notation,

$$\frac{\partial^2\phi^\alpha}{\partial x^i\,\partial x^j}=X_{ij}^\alpha\left(x^k,\phi^\beta,\frac{\partial\phi^\beta}{\partial x^k}\right). \tag{24}$$

We shall demonstrate the formal similarity between the theory of second-order jet fields and the theory of second-order differential equation fields by proving three theorems. The theorems all involve the vertical vector-valued m-form S_Ω on $J^1\pi$, defined intrinsically in Saunders (1987) and expressed in coordinates as $\partial/\partial u_i^\alpha\otimes(\mathrm{d}u^\alpha-u_j^\alpha\,\mathrm{d}x^j)\wedge(\partial/\partial x^i\lrcorner\Omega)$.

The first two theorems both concern Lagrangian field theories, so we suppose given a Lagrangian function $L\colon J^1\pi\to\boldsymbol{R}$. An extremal of L is a section ϕ of π having the property that, for every vertical vector field Γ on E,

$$\int(j^1\phi)^*\Gamma^1(L)\Omega=0 \tag{25}$$

where Γ^1 is the prolongation of Γ to $J^1\pi$. In the same way as for mechanics, one finds that ϕ satisfies the Euler–Lagrange equations:

$$(j^2\phi)^*\left(\frac{\partial L}{\partial u^\alpha}-\frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha}\right)=0 \tag{26}$$

(see, for example, Krupka 1983). Furthermore, the Cartan form for L is the m-form Θ_L on $J^1\pi$ defined by $\Theta_L=S_\Omega(\mathrm{d}L)+L\Omega$ (see Goldschmidt and Sternberg (1973) for an alternative intrinsic description of this object).

Theorem 4.3. Let X be an integrable second-order jet field. Then the integral sections of X are extremals of L if, and only if,

$$i_{\tilde{X}}\, \mathrm{d}\Theta_L = (m-1)\, \mathrm{d}\Theta_L. \tag{27}$$

Proof. If ϕ is an integral section of X then $j^2\phi = X \circ j^1\phi$, so the Euler–Lagrange equations become

$$(j^1\phi)^* X^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i} \frac{\partial L}{\partial u_i^\alpha} \right) = 0 \tag{28}$$

giving, explicitly,

$$(j^1\phi)^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i\, \partial u_i^\alpha} - u_i^\beta \frac{\partial^2 L}{\partial u^\beta\, \partial u_i^\alpha} - X_{ij}^\beta \frac{\partial^2 L}{\partial u_j^\beta\, \partial u_i^\alpha} \right) = 0. \tag{29}$$

Since there is an integral section through each point of $J^1\pi$ we obtain

$$\frac{\partial^2 L}{\partial u_j^\beta\, \partial u_i^\alpha} X_{ij}^\beta = \frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i\, \partial u_i^\alpha} - u_i^\beta \frac{\partial^2 L}{\partial u^\beta\, \partial u_i^\alpha}. \tag{30}$$

A straightforward calculation then shows that $i_{\tilde{X}}\, \mathrm{d}\Theta_L = (m-1)\, \mathrm{d}\Theta_L$.

Conversely, if $\tilde{X}$ satisfies this relationship then the coordinates X_{ij}^β must satisfy equation (30), so that

$$X^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i} \frac{\partial L}{\partial u_i^\alpha} \right) = 0 \tag{31}$$

on $J^1\pi$. Then, if ϕ is any integral section of X,

$$(j^2\phi)^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u_i^\alpha} \right) = (j^1\phi)^* X^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u_i^\alpha} \right) = 0. \tag{32}$$

We call such a jet field X an Euler–Lagrange field for the Lagrangian L. The prolongations $j^1\phi$ of the extremals ϕ of L then describe the integral manifolds of the horizontal distribution generated by the tensor field $\tilde{X}$.

Theorem 4.3 corresponds to the result from time-dependent mechanics that the Euler–Lagrange vector field is a characteristic of the Cartan 2-form. By contrast, our second (closely related) theorem has the flavour of autonomous mechanics in that it uses the $(m+1)$-form $\omega_L = \mathrm{d}_v(S_\Omega\, \mathrm{d}L)$ defined in a way which mimics the definition of the 2-form $\omega_L = \mathrm{d}(S(\mathrm{d}L))$ in tangent bundle geometry (Crampin 1983). This theorem associates the tensor field $\tilde{X}$ to the Euler–Lagrange $(m+1)$-form δL defined by

$$\delta L = \mathrm{d}(L\Omega) + \mathrm{d}_h \Theta_L. \tag{33}$$

In coordinates, $\delta L = (\partial L/\partial u^\alpha - (\mathrm{d}/\mathrm{d}x^i)\, \partial L/\partial u_i^\alpha)\, \mathrm{d}u^\alpha \wedge \Omega$ and so $X^*\delta L = 0$; X may be regarded as a mapping which replaces the second-derivative coordinates in the expression for δL with values which satisfy the Euler–Lagrange equations. Consequently δL vanishes along $j^2\phi$ for any extremal ϕ of L.

Theorem 4.4. If X is an Euler–Lagrange field for L then $i_{\tilde{X}}\omega_L = (m-1)\omega_L + \delta L$.

Proof. By calculation.

Our final theorem is concerned with arbitrary second-order jet fields, not just those arising from Lagrangian functions. The corresponding result from tangent-bundle geometry is that each second-order differential equation field $\Lambda: TE \to TTE$ determines a decomposition of the bundle $\tau_{TE}: TTE \to TE$ as a direct sum $(V\tau_E \to TE) \oplus (H_\Lambda \to TE)$. The proof involves taking the Lie derivative by Λ of the vertical endomorphism S and observing that $Q = \frac{1}{2}(I - \mathscr{L}_\Lambda S)$ is a projection operator whose kernel is the set of vertical vectors. The image of Q therefore determines a set of horizontal vectors (Crampin 1983).

Theorem 4.5. Each second-order jet field X on $J^1\pi$ determines a decomposition of the bundle $V\pi_1 \to J^1\pi$ as a direct sum $(V\pi_{1,0} \to J^1\pi) \oplus (H_X \to J^1\pi)$, i.e. every tangent vector to $J^1\pi$ vertical over M is assigned a unique component which is vertical over E.

Proof. In the present context, the analogue of the Lie derivative is the Frölicher-Nijenhuis bracket of vector-valued forms described in § 2. We therefore consider the vector-valued $(m+1)$-form $[S_\Omega, \tilde{X}]$. Regarding this as an operator from 1-forms to $(m+1)$-forms we have, for every 1-form σ on $J^1\pi$, $[S_\Omega, \tilde{X}](\sigma) \in \Lambda_1^{m+1}(\pi_1)$ and if $\sigma \in \Lambda_0^1(\pi_1)$ then $[S_\Omega, \tilde{X}](\sigma) = 0$; this may be seen from the coordinate representation of $[S_\Omega, \tilde{X}]$:

$$[S_\Omega, \tilde{X}] = \frac{\partial}{\partial u_i^\alpha} \otimes (\mathrm{d}u_i^\alpha \wedge \Omega) - \left(\frac{\partial}{\partial u^\alpha} + \frac{\partial X_{ij}^\beta}{\partial u_i^\alpha} \frac{\partial}{\partial u_j^\beta}\right) \otimes (\mathrm{d}u^\alpha \wedge \Omega). \tag{34}$$

If we write $I \wedge \Omega$ for the vector-valued $(m+1)$-form defined by $I \wedge \Omega(\sigma) = \sigma \wedge \Omega$ and put $Q = \frac{1}{2}(I \wedge \Omega - [S_\Omega, \tilde{X}])$ then again $Q(\sigma) \in \Lambda_1^{m+1}(\pi_1)$ and if $\sigma \in \Lambda_0^1(\pi_1)$ then $Q(\sigma) = 0$; in fact

$$Q = \left(\frac{\partial}{\partial u^\alpha} + \frac{1}{2} \frac{\partial X_{ij}^\beta}{\partial u_i^\alpha} \frac{\partial}{\partial u_j^\beta}\right) \otimes (\mathrm{d}u^\alpha \wedge \Omega). \tag{35}$$

We now use the fact that there is a canonical isomorphism between $V^*\pi_1$ (the dual of $V\pi_1$) and $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$. The vector-valued $(m+1)$-form Q defines a mapping (also called Q) from $T^*J^1\pi$ to $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ by the rule $Q(\sigma_p) = Q(\sigma)_p$ where $\sigma_p \in T_p^*J^1\pi$; if it so happens that $\sigma_p \in \pi_1^*T^*M$ then $Q(\sigma_p) = 0$. However, each $\theta_p \in T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ has a representative σ_p satisfying $\theta_p = \sigma_p \wedge \Omega_p$, and any two such representatives differ by an element of $\pi_1^*T^*M$. We may therefore define $Q(\theta_p)$ to equal $Q(\sigma_p)$ where σ_p is a representative of θ_p. The resulting endomorphism of $T^*J^1\pi \wedge \pi_1^*\{\Omega\}$ (and hence of $V^*\pi_1$) yields the dual endomorphism of $V\pi_1$ which is a projection operator expressed in coordinates as

$$\xi^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_p + \eta_i^\alpha \left.\frac{\partial}{\partial u_i^\alpha}\right|_p \mapsto \xi^\alpha \left(\left.\frac{\partial}{\partial u^\alpha}\right|_p + \frac{1}{2} \left.\frac{\partial X_{ij}^\beta}{\partial u_i^\alpha}\right|_p \left.\frac{\partial}{\partial u_j^\beta}\right|_p \right). \tag{36}$$

The kernel of this endomorphism is $V\pi_{1,0}$, and defining its image to be H_X gives the required decomposition of $V\pi_1$.

5. Conclusion

The theorems proved in the preceding section demonstrate that several of the constructions used in the theory of mechanics are actually special cases of more general constructions which may be applied to field theories as well. Of course, calculations with a tensor field are a little more complicated than those involving merely a vector field; however, the idea of a jet field and its associated tensor is so natural that one may expect many more of the existing results from the geometric theory of second-order differential equations to be established in this more general setting.

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